INDUCTIVE THEORIES AND THEIR FORCING COMPANIONS

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ABSTRACT

A decomposition theorem for ideals of a distributive lattice is related to a classification of the generic models of an arbitrary inductive theory, generalizing, for example, the classification of algebraically closed fields according to their characteristics.

Introduction

Let T be a deductively closed theory in the Lower Predicate Calculus (i.e., in a language $L_{\omega\omega}$), and let T_U be the set of universal sentences which belong to T. If T is the deductive closure of T_U, then we call T *universal.* We shall be concerned, in the first place, with universal theories T.

T_U is a subset of U, where U is the set of universal sentences in $L_{\omega\omega}$. Passing to the Lindenbaum algebra $\mathscr{L}$ of $L_{\omega\omega}$, we see that T_U corresponds to an ideal in the sublattice $\mathscr{L}(U)$ of (logical-equivalence classes of) universal sentences. It is shown in [7], within a more general framework, that any such ideal can be represented as the intersection of irreducibles, or prime ideals, and that this representation, subject to certain conditions, is unique. To this representation there corresponds a decomposition of the class or variety, V, of models of T_U into a number of subvarieties, V_ν. To V and to each V_ν, there corresponds a subclass, its class of generic structures. For example, if T is the theory of commutative integral dcmains (formalized for convenience as a universal theory), then the classes V_ν are just the

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classes of commutative integral domains of specified characteristic, and the class V is their union. At the same time, the generic structures within each V_ν are the algebraically closed fields of the appropriate characteristics, and their union is just the class of generic structures of V. *It is main purpose of the present paper to show that this is also the situation for a general universal theory* T *in* $L_{\omega\omega}$.

We shall assume familiarity with [11] or with earlier work on forcing. Thus, we adapt the work of [7] (to the extent to which it is required here) to contemporary terminology. As for forcing we prove the necessary results ab initio, except for the existence theorem for generic structures in infinite forcing (see [10, 4, or 2]).

1. Distributive lattices and Boolean algebras

We will use the term *distributive lattice* $\mathscr{L} = \langle L, \vee, \wedge, 0, 1\rangle$ for a distributive lattice in the usual sense with 0 and 1 as distinct least and greatest elements. Given $a, b \in L$ we write "$a < b$" for "$a \wedge b = a$". An ideal I of $\mathscr{L}$ is a non-empty subset of L such that for all $a, b \in L$ (i) $a \in I$ and $a < b$ implies $b \in I$, and (ii) $a, b \in I$ implies $a \wedge b \in I$. Note that our definition of ideal corresponds to the notion of a dual ideal or filter in a Boolean ring; nevertheless, even if $\mathscr{L}$ is a Boolean algebra we will use the term ideal in the above sense.* An ideal I will be called *proper* if $I \neq L$ (i.e., $0 \notin I$), otherwise *improper*; I will be called *trivial* if $I = \{1\}$.

Clearly an intersection of ideals is again an ideal. Thus we can speak of the ideal generated by an arbitrary subset of L. The ideals of $\mathscr{L}$ form a complete lattice relative to the partial order of set-theoretic inclusion; moreover, given two ideals I and J of $\mathscr{L}$, an easy application of the distributive laws shows that the inf of I and J is given by:

$$I \wedge J = I \cap J = \{a \vee b,\ a \in I,\ b \in J\},$$

while the sup of I and J is given by:

$$I \vee J = \{a \wedge b : a \in I,\ b \in J\}.$$

Finally the sup of a directed system $\mathscr{S}$ of ideals of $\mathscr{L}$ is just $\bigcup \mathscr{S}$.

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* We have used the term "ideal" rather than "dual ideal" or "filter" because this is what these entities become when we specialize to classical algebra. Readers who object to our terminology are advised to turn the pages upside down.

THEOREM 1.1. *The complete lattice of any distributive lattice is distributive. In fact, if $\mathscr{S}_1$ and $\mathscr{S}_2$ are families of ideals, then*

$$(\bigvee \mathscr{S}_1) \wedge (\bigvee \mathscr{S}_2) = \bigvee\{I \wedge J\colon\ I \in \mathscr{S}_1,\ J \in \mathscr{S}_2\}.$$

PROOF. It suffices to show that if I is an ideal and $\mathscr{S}$ is a family of ideals, then

$$I \wedge (\bigvee \mathscr{S}) = \bigvee\{I \wedge J\colon\ J \in \mathscr{S}\},$$

since the law of the theorem follows by a double application of this formula, and the two dual distributive laws are consequences of each other.

Clearly the right-hand-side is contained in the left-hand-side. On the other hand if $a \in I \cap (\bigvee \mathscr{S})$, then $a = a_1 \wedge \cdots \wedge a_n$, where each a_i belongs to a member of $\mathscr{S}$. Evidently each a_i belongs to I, and thus each a_i belongs to an ideal of the form $I \cap J$ with $J \in \mathscr{S}$. Finally each b_i, and hence a, belongs to $\bigvee\{I \wedge J\colon\ J \in \mathscr{S}\}$. This completes our proof.

An element a of a lattice $\mathscr{L}$ is called *irreducible* if for any $b, c \in L$, $b \wedge c = a$ implies $b = a$ or $c = a$. Note that, in a distributive lattice, a is irreducible if and only if for $b, c \in L$, $b \wedge c < a$ implies $b < a$ or $c < a$. We shall see that in the lattice of ideals of a distributive lattice there is a rich supply of irreducible elements; this need not be the case in a distributive lattice in general. Note that an ideal I is irreducible if and only if for all $a, b \in L$, $a \vee b \in I$ implies $a \in I$ or $b \in I$. Irreducible ideals (and in Boolean algebras, prime ideals) can be improper while maximal ideals are necessarily proper.

If $\mathscr{L}_1$ is a sublattice of $\mathscr{L}_2$ and I is an ideal of $\mathscr{L}_2$, then $I \cap L_1$ is an ideal of $\mathscr{L}_1$ which is proper if I is proper and irreducible if I is irreducible. Conversely, given an ideal J of $\mathscr{L}_1$, the ideal I generated by J in $\mathscr{L}_2$ is given by:

$$I = \{a \in L_2\colon \text{there exists } b \in J \text{ such that } b < a\}.$$

Clearly $I \cap L_1 = J$, and I is proper i^f J is; however, I need not be irreducible if J is. Nevertheless, we have the following.

THEOREM 1.2. *If $\mathscr{L}_1$ is a sublattice of distributive lattice $\mathscr{L}_2$ and J is an irreducible ideal of $\mathscr{L}_1$, then there exists an irreducible ideal I of $\mathscr{L}_2$ such that $I \cap L_1 = J$.*

PROOF. The theorem holds for finitely generated distributive lattices, since in this case $\mathscr{L}_1$ and $\mathscr{L}_2$ are finite, all ideals are principal, and the proper irreducible ideals are exactly the principal ideals generated by atoms. Thus by the compactness theorem, the following theory is consistent:

$$T$$

$$D(\mathscr{L}_2)$$

$$I(a),\ a \in J$$

$$\neg I(a),\ a \in L_1 - J$$

$$\forall x, y([I(x) \wedge x < y \rightarrow I(y)]$$

$$\wedge [I(x) \wedge I(y) \rightarrow I(x \wedge y)]$$

$$\wedge [I(x \wedge y) \rightarrow I(x) \vee I(y)]),$$

where we have identified the members of L_2 with constants denoting them, $D(\mathscr{L}_2)$ stands for the diagram of $\mathscr{L}_2$, I is a unary relation symbol, and T is the theory of distributive lattices. A model of this theory gives us an irreducible ideal I' in a distributive lattice $\mathscr{L}_3$ which extends $\mathscr{L}_2$ such that $I' \cap L_1 = J$. Clearly, $I = I' \cap L_2$ is the desired irreducible ideal of $\mathscr{L}_2$, and our proof is complete.

We remark that the proof of theorem 1.2 can be turned into a similar proof that every distributive lattice is imbeddable in a Boolean algebra. Such a proof avoids a direct use of Zorn's lemma through an appeal to the compactness theorem. On the other hand, the following result seems to require a full-fledged use of the Axiom of Choice.

If $a < b$ are elements of a distributive lattice $\mathscr{L}$ with b irreducible, we say that b is *minimal irreducible over* a or a *component of* a if for all $c \in L$, $a < c < b$ and c irreducible implies $b = c$.

LEMMA 1.3. *Given elements $a < b$ of a complete distributive lattice $\mathscr{L}$ with b irreducible, there exists a minimal irreducible c over a such that $c < b$.*

PROOF. Let S be the set of irreducibles which contain a and are contained in b. Let $C \subset S$ be a descending chain. If $c = \bigwedge C$ were not irreducible, there would exist b_1 and b_2 such that $b_1 \wedge b_2 = c$, $b_1 \neq c$, and $b_2 \neq c$. Then for some $c_1 \in C$. $b_1 \nless c_1$, and for some $c_2 \in C$, $b_2 \nless c_2$. But since C is a chain, $c_1 \wedge c_2 \in C$, and we have $b_1 \wedge b_2 = c < c_1 \wedge c_2$, $b_1 \nless c_1 \wedge c_2$, and $b_2 \nless c_1 \wedge c_2$, contradicting the irreduciblity of $c_1 \wedge c_2$, Finally, since each descending chain in S has an inf in S, there exist minimal elements of S by Zorn's lemma.

THEOREM 1.4. *Every ideal in a distributive lattice is the intersection of its* *components. An ideal I contains an ideal J if and only if every component of I contains J.*

PROOF. Let I be an ideal of a distributive lattice $\mathscr{L}$. Imbed $\mathscr{L}$ in a Boolean algebra $\mathscr{B}$. Note that the irreducible ideals of $\mathscr{B}$ are just the prime ideals. Now given an element a of $L-I$, there exists a prime ideal I' of $\mathscr{B}$ such that $I \subset I'$ and $a \notin I'$. Then $I' \cap L$ is an irreducible ideal of $\mathscr{L}$ containing I and to which a does not belong. By Lemma 1.3 there exists a component of I contained in $I' \cap L$. Since a was arbitrary, I is the intersection of its components.

The second part of the theorem is immediate.

Given a family $\mathscr{S}$ of ideals it is easy to see that the irreducible ideals containing $\bigvee \mathscr{S}$ are exactly those which contain each member of $\mathscr{S}$. On the other hand, it is not generally the case that the irreducible ideals containing $\bigwedge \mathscr{S}$ are exactly those which contain some member $\mathscr{S}$. However we have the following result.

THEOREM 1.5. *Given a family $\mathscr{S}$ of ideals the following are equivalent:*

(i) *every irreducible ideal containing $\bigwedge \mathscr{S}$ contains some member of $\mathscr{S}$;*

(ii) *every component of $\bigwedge \mathscr{S}$ is a component of some member of $\mathscr{S}$.*

PROOF. Trivial.

If the conditions in Theorem 1.5 apply to a family $\mathscr{S}$ of ideals we say that $\mathscr{S}$ is *regular*. Clearly the set of components of a given ideal is regular, and every finite family of ideals is regular. We will give an example of an irregular family of ideals in the next section.

THEOREM 1.6. *Let $I_{\xi,\eta}$, $\langle \xi,\eta\rangle \in \alpha \times \beta$, be a doubly indexed family of ideals in a distributive lattice such that, for each $\xi \in \alpha$, $\{I_{\xi,\eta}: \eta \in \beta\}$ is regular. Then*

$$\mathscr{S} = \{ \bigvee_{\xi \in \alpha} I_{\xi\, f(\xi)} : f \in \beta^{\alpha}\}$$

is regular, and

$$\bigvee_{\xi \in \alpha} \bigwedge_{\eta \in \beta} I_{\xi,\eta} = \bigwedge_{f \in \beta}{}_{\alpha} \bigvee_{\xi \in \alpha} I_{\xi, f(\xi)}$$

PROOF. Let $I = \bigvee_{\xi \in \alpha} \bigwedge_{\eta \in \beta} I_{\xi,\eta}$. Clearly $I \subset \bigwedge_{f \in \beta^{\alpha}} \bigvee_{\xi \in \alpha} I_{\xi,f(\xi)}$. We can demontrate the regularity of $\mathscr{S}$ and prove the containment in the opposite direction simultaneously by showing that every irreducible ideal which contains I also contains $\bigvee_{\xi \in \alpha} I_{\xi,f(\xi)}$ for some $f \in \beta^{\alpha}$. Thus, let $I' \supset I$ be irreducible. Then for each $\xi \in \alpha$, $I' \supset \bigwedge_{\eta \in \beta} I_{\xi,\eta}$, and by the regularity of $\{I_{\xi,\eta}; \eta \in \beta\}$, there exists $\eta \in \beta$ such that $I' \supset I_{\xi,\eta}$. Let $f \in \beta^{\alpha}$ choose such an η for each ξ. Then $I' \supset \bigvee_{\xi \in \alpha} I_{\zeta,f(\xi)}$, and our proof is complete.

The following facts are easily checked and left to the reader. A proper ideal in a Boolean algebra is irreducible if and only if it is maximal. There is a dual isomorphism between the ideals of a Boolean algebra and the closed sets of the corresponding Stone space. An intersection of ideals corresponds to the closure of the union of the corresponding closed sets. A family of ideals is regular if and only if the union of the corresponding family of closed sets is closed.

THEOREM 1.7. *Let $\mathscr{L}$ be a sublattice of a Boolean algebra $\mathscr{B}$, and let I be an ideal of $\mathscr{L}$. For each element a of $\mathscr{B}$, let a' be the complement of a. Then I is proper and irreducible if and only if $I \cup \{a' : a \in L - I\}$ generates a proper ideal in $\mathscr{B}$.*

Let $J \supset I$ be irreducible. Then the following are equivalent:

(i) *J is a component of I,*

(ii) *every prime ideal (in $\mathscr{B}$) which contains $I \cup \{a' : a \in L - J\}$ contains J,*

(iii) *every maximal ideal (in $\mathscr{B}$) which contains $I \cup \{a' : a \varepsilon L - J\}$ contains J,*

(iv) *the ideal generated by $I \cup \{a' : a \in L - J\}$ contains J.*

PROOF. For the first part we note that I is proper and irreducible iff there exists a maximal ideal I' in $\mathscr{B}$ such that $I = I' \cap L$ iff there exists a proper ideal I' of $\mathscr{B}$ such that $I' \supset I \cup \{a' : a \in L - I\}$.

(i) $\Rightarrow$ (ii): Suppose that J is a component of I and that $P \supset I \cup \{a' : a \in L - J\}$ is a prime ideal of $\mathscr{L}$. Then $P \cap L$ is clearly an irreducible ideal of $\mathscr{L}$ containing I and contained in J. Thus, since J is a component, $P \cap L = J$.

(ii) $\Rightarrow$ (iv): This is clear, since every ideal is the intersection of the irreducible ideals containing it.

(iv) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): Suppose that J' is an irreducible ideal of $\mathscr{L}$ such that $I \subset J' \subset J$. Then $J' = P \cap L$ for some prime ideal of $\mathscr{B}$. If $P = B$, then $J' = L$ and $J = J'$. Otherwise, P is maximal and contains $I \cup \{a' : a \in L - J\}$. By assumption $P \supset J$ so that $J = J'$. Thus in either case J is a component, and our proof is complete.

2. Theories and varieties of structures

We will now apply the general development of the previous section to the study of first order theories and their models. Let L be a fixed vocabulary, and let $\mathscr{L}$ be the corresponding Lindenbaum algebra of logical-equivalence classes of sentences of the language $L_{\omega\omega}$. Let $\mathscr{L}_U$ be the sublattice of (equivalence classes of) universal sentences, and let $\mathscr{L}_I$ be the sublattice of inductive (i.e., $\forall\exists$) sentences.

A (deductively closed) theory T in $L_{\omega\omega}$ corresponds to a proper ideal in $\mathscr{L}$. Hereafter we will work modulo logical equivalence, and therefore identify sentences with members of $\mathscr{L}$ and theories with proper ideals of $\mathscr{L}$.

Given a theory T, we let T_U be $T \cap L_U$ and $T_I = T \cap L_I$. Given a proper ideal J in a sublattice of $\mathscr{L}$, we let $\bar{J}$ be the ideal generated by J in $\mathscr{L}$. A theory is called *universal* if $T = T_U$ and *inductive* if $T = T_I$.

Given a theory T, we let $\mathscr{V}_T$, the *Stone variety* of T, be the set of maximal ideals which extend T. This is just the set of elementary equivalence types of models of T. We will henceforth say that a structure $\mathscr{M}$ belongs to $\mathscr{V}_T$ if its complete theory does.

An *embedding* of a structure $\mathscr{M}$ in a structure $\mathscr{N}$ is an isomorphism of $\mathscr{M}$ onto a substructure of $\mathscr{N}$. A theory is said to have the *joint-embedding property* if any two models can be embedded in a third model. The following characterization of theories with joint-embedding is well-known [8].

THEOREM 2.1. *A theory T has the joint-embedding property if and only if T_U is irreducible.*

PROOF. We first suppose that T has joint-embedding. Let σ_1 and σ_2 be universal sentences such that $\sigma_1 \notin T_U$ and $\sigma_2 \notin T_U$. Let $\mathscr{M}_1$ be a model of $T \cup \{\neg\sigma_1\}$ and $\mathscr{M}_2$ be model of $T \cup \{\neg\sigma_2\}$. If both are embedded in $\mathscr{M}_3$, then we have

$$\mathscr{M}_3 \models T \cup \{\neg\sigma_1, \neg\sigma_2\},$$

since existential sentences are preserved by embeddings. In other words, we have established that $\sigma_1 \in L_U$, $\sigma_2 \in L_U$, and $\sigma_1 \vee \sigma_2 \in T_U$ implies $\sigma_1 \in T_U$ or $\sigma_2 \in T_U$; i.e., T_U is irreducible.

Conversely, if T_U is irreducible and $\mathscr{M}_1$ and $\mathscr{M}_2$ are models of T, one can check that the diagrams are mutually consistent with T through an appeal to the first part of Theorem 1.7.

THEOREM 2.2. *A set of theories $\mathscr{S}$ is regular if and only if every model of $\cap\mathscr{S}$ is a model of some member of $\mathscr{S}$, in other words, $\mathscr{V}_{\cap\mathscr{S}} = \cup\{\mathscr{V}_T\colon T \in \mathscr{S}\}$.*

PROOF. Trivial.

It is now easy to check that if T_p is the theory of algebraically closed fields of characteristic p, then $\mathscr{S} = \{T_p\colon p \text{ prime}\}$ is not regular since $\cap\mathscr{S}$ is the theory of all algebraically closed fields, including those of characteristic zero.

We say that a theory T_1 is *model-consistent* with another theory T_2 if every model of T_2 is embeddable in a model of T_1. A standard application of diagrams establishes the following theorem [8].

THEOREM 2.3. *Given theories T_1 and T_2, T_1 is model-consistent with T_2 if and only if $T_{1U} \subset T_{2U}$.*

3. Existentially complete models and generic models

Let $\mathcal{M} \subset \mathcal{N}$ be arbitrary structures. We sat that $\mathcal{M}$ *is existentially closed in* $\mathcal{N}$ if for every existential formula ϕ and every assignment $\boldsymbol{a}$ of members of M to the free variables of ϕ, $\mathcal{M} \models \phi[\boldsymbol{a}]$ iff $\mathcal{N} \models \phi[\boldsymbol{a}]$. Given a theory T, we say that a model $\mathcal{M}$ of T is *existentially complete* if $\mathcal{M}$ is existentially closed in every extension which is a model of T. Every inductive theory T has a rich supply of existentially complete models. In fact there exists a class of models, which we call the *generic models*, characterizable (as a class) in the following way [11]

(i) every generic model is a model of T;
(ii) every model of T can be embedded in a generic model;
(iii) if $\mathcal{M}_1 \subset \mathcal{M}_2$ are generic, then $\mathcal{M}_1 \prec \mathcal{M}_2$;
(iv) if $\mathcal{M}_1 \prec \mathcal{M}_2$ and $\mathcal{M}_2$ is generic, then $\mathcal{M}_1$ is generic.

It is easy to check that every generic model is existentially complete. We prove a strong form of uniqueness for such a class in the next theorem.

THEOREM 3.1. *Given an inductive theory T, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of structures satisfying:*

(i′) *every of $\mathcal{K}_i$ can be embedded in a model of T;*
(ii) *every model of T can be embedded in a member of $\mathcal{K}_i$;*
(iii) *if $\mathcal{M}_1 \subset \mathcal{M}_2$ are members of $\mathcal{K}_i$, then $\mathcal{M}_1 \prec \mathcal{M}_2$;*
(iv) *if $\mathcal{M}_1 \prec \mathcal{M}_2$ and $\mathcal{M}_2 \in \mathcal{K}_i$, then $\mathcal{M}_1 \in \mathcal{K}_i$. Then $\mathcal{K}_1 = \mathcal{K}_2$.*

PROOF. Let $\mathcal{M} = \mathcal{M}_0 \in \mathcal{K}_1$. By (i′) for $\mathcal{K}_1$ and (ii) for $\mathcal{K}_2$, there exists $\mathcal{N}_0 \in \mathcal{K}_2$ such that $\mathcal{M}_0 \subset \mathcal{N}_0$. Similarly, there exists $\mathcal{M}_1 \in \mathcal{K}_1$ with $\mathcal{N}_0 \subset \mathcal{M}_1$. Continuing in this fashion we obtain a chain:

$$\mathcal{M}_0 \subset \mathcal{N}_0 \subset \mathcal{M}_1 \subset \mathcal{N}_1 \subset \cdots,$$

where each $\mathcal{M}_i \in \mathcal{K}_1$ and each $\mathcal{N}_i \in \mathcal{K}_2$. The union $\mathcal{M}_\omega = \mathcal{N}_\omega$ of this chain is an elementary extension of $\mathcal{M}_0$ and $\mathcal{N}_0$ by assumption (iii) for the two classes. By (iv) for $\mathcal{K}_2$, $\mathcal{M} \in \mathcal{K}_2$, and we have shown that $\mathcal{K}_1 \subset \mathcal{K}_2$. By symmetry $\mathcal{K}_2 \subset \mathcal{K}_1$, so that $\mathcal{K}_1 = \mathcal{K}_2$ as desired.

THEOREM 3.2. *The class of generic models of an inductive theory T is inductive; i.e., a union of a chain of generic models is generic.*

PROOF. A union of a chain of generic models, being a model of T, is embeddable in a generic model $\mathscr{M}$ of T. Since each of the terms is an elementary substructure of $\mathscr{M}$ (by (iii)), the union is an elementary substructure of $\mathscr{M}$ and is generic by (iv).

Let us define T^F to be the theory of the generic models of T, and let T^I be $T^F \cap L_I$. We can characterize T^I in somewhat different terms.

THEOREM 3.3. *T^I is the set of all sentences σ in L_I such that $T \cup \{\sigma\}$ is model-consistent with T, and it is also the set of all sentences in L_I which hold in all existentially complete models of T.*

PROOF. If $\sigma \in T^I$, then $T \cup \{\sigma\}$ is clearly model-consistent with T. On the other hand, if $\sigma \in L_I$ is such that $T \cup \{\sigma\}$ is model-consistent with T and if $\mathscr{M}$ is an existentially complete model of T, then we can embed $\mathscr{M}$ in a model $\mathscr{N}$ of $T \cup \{\sigma\}$. By the definition of an existentially complete model, each instantiation of σ holds in $\mathscr{M}$. Lastly, if a sentence $\sigma \in L_I$ holds in all existentially complete models, it holds in all generic models, i.e., belongs to T^I.

From the characterization of T^I given above, we see that it can be described as *the* maximal model-consistent, inductive extension of T. Kaiser [5] has called this the *inductive hull* of T.

In general, not all existentially complete models of T are generic. However, we have the following theorem.

THEOREM 3.4. *Let T be an inductive theory. Then the following are equivalent:*

(i) *the generic models of T form an elementary class;*

(ii) *the existentially complete models of T form an elementary class;*

(iii) *there exists a model-complete extension T^* of T which is model-consistent with T (and the generic as well as existentially complete models of T are precisely the models of T^*).*

Each implies that the generic and existentially complete models are the same.

Proof. (i) $\Rightarrow$ (ii): If the generic models form an elementary class, it is model-complete, hence inductive, and $T^F = (T^I)$. (ii) follows from Theorem 3.3.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i): It follows from the primitive sentence test for model-completeness and Theorem 3.3, that the desired T^* in (iii) can be taken to be

T^I, and that furthermore the models of T^* are then exactly the generic models of T.

We remark that the theory T^* in (iii) above, when it exists, is called the model-companion of T [1]. Eklof and Sabbagh [3] showed the equivalence of (ii) and (iii) and proved that the theory of groups has no model-companion. Macintyre [6] then showed that there are existentially-complete groups which are elementarily inequivalent to the generic groups—his proof uses the notion of generic models in the sense of *finite* forcing [9, 1]. Eklof and Sabbagh [3] also showed that the theory of modules over a fixed ring has a model-companion if and only if the ring is coherent. The first author (unpublished) has recently shown that the three classes of existentially-complete, generic, and finitely generic modules over a fixed ring always coincide, i.e., even when a model-companion does not exist.

THEOREM 3.5. *Let T_1 and T_2 be inductive theories. Then the following are equivalent*:

(i) *T_1 and T_2 are mutually model-consistent*;

(ii) $T_{1U} = T_{2U}$;

(iii) *T_1 and T_2 have the same generic models*;

(iv) *T_1 and T_2 have the same existentially complete models*;

(v) $T_1^F = T_2^F$

(vi) $T_1^I = T_2^I$.

PROOF. (i) $\Leftrightarrow$ (ii) is immediate from Theorem 2.3.

(i) $\Rightarrow$ (iii) follows readily from Theorem 3.1.

(iii) $\Rightarrow$ (iv) follows from the observation that the existentially-complete models of an inductive theory are precisely the existentially closed substructures of generic models.

(iii) $\Rightarrow$ (v), (iv) $\Rightarrow$ (vi), and (v) $\Rightarrow$ (vi) are immediate.

(vi) $\Rightarrow$ (i) is a direct consequence of the remark that every inductive theory is mutually model consistent with its inductive hull together with the transitivity of the relation of model-consistency.

COROLLARY 3.6. *Let T be an inductive theory. Then*

(i) $T^F \cap L_U = T^I \cap L_U = T_U$, and

(ii) $T^F = \overline{(T^I)}^F = \overline{(T_U)}^F$.

PROOF. Trivial.

We are now in a position to illuminate in model-theoretic terms the irreducibility of a universal theory as an ideal of $\mathscr{L}_U$.

THEOREM 3.7. *Let T be an inductive theory. Then the following are equivalent* [11]:

(i) *T_U is irreducible in $\mathscr{L}_U$;*
(ii) *T has joint embedding;*
(iii) *T^I has joint embedding;*
(iv) *T^F has joint embedding;*
(v) *any two generic models of T can be embedded in a third;*
(vi) *T^F is complete.*

PROOF. By Theorem 2.1 above, any theory T has joint-embedding if and only if T_U is irreducible. Since by Corollary 3.6 $T_U = (T^I)_U = (T^F)_U$, the equivalence of (i), (ii), (iii) and (iv) is immediate.

(ii) $\Rightarrow$ (v): Any two generic models of T can be embedded in common extension which is a model of T, and the latter can be embedded in a generic model.

(v) $\Rightarrow$ (vi): It suffices to show that any two generic models are elementarily equivalent, but (v) implies that any two have a common elementary extension.

(vi) $\Rightarrow$ (i): Any complete theory, being a prime ideal, is irreducible in $\mathscr{L}$. Thus, its intersection with any sublattice is irreducible. But $T^F \cap L_U = T_U$.

THEOREM 3.8. *Let T be an inductive theory, and let J be an irreducible ideal of $\mathscr{L}_U$. If J is a component of T_U, then the generic models of $\bar{J}$ are precisely the generic models of T which are models of $\bar{J}$. Conversely, if there exists a generic model of T which is also a generic model of $\bar{J}$, then J is a component of T_U.*

PROOF. Suppose that J is a component of T_U, and let $\mathscr{K}$ be the class of generic models of T which are models of $\bar{J}$. We must show that $\mathscr{K}$ is the class of generic models of $\bar{J}$. Conditions (i), (iii), and (iv) of the characterization are immediate. To verify (ii), let $\mathscr{M}$ be a model of $\bar{J}$. We may embed $\mathscr{M}$ in a generic model $\mathscr{M}'$ of $\bar{J}$. Since J is irreducible, we see by Theorem 3.7 that $\bar{J}^F$ is complete and by Corollary 3.6 that $\bar{J}^F \cap L_U = J$, so that $\mathscr{M}'$ is a model of $J \cup \{a' : a \in L_U - J\}$. We can now embed $\mathscr{M}'$ in a generic model $\mathscr{N}$ of T by Theorem 2.3. $\mathscr{N}$ is now a model of $T_U \cup \{a' : a \in L_U - J\}$, since existential sentences persist upwards, and by Theorem 1.7 ((i) $\Rightarrow$ (iv)) we conclude that $\mathscr{N}$ is a model of $\bar{J}$. Thus $\mathscr{M} \subset \mathscr{N} \in \mathscr{K}$, and we have shown that condition (ii) holds, as desired.

For the converse, let J be irreducible, and suppose that $\mathcal{M}$ is a generic model of T which is also generic for $\bar{J}$. If $J' \subset J$ is a component of T_U, then by what we have just shown, we see that $\mathcal{M}$ is a generic model of $\bar{J}'$. But $\bar{J}'^F$ is complete, and thus $J' = \bar{J}'^F \cap L_U$ is the ideal of *all* universal sentences which hold in $\mathcal{M}$. We conclude (since $\mathcal{M}$ is a model of $\bar{J}$) that $J \subset J'$, and thus $J = J'$. This shows that J is a component of T_U.

THEOREM 3.9. *Let T be inductive. Then the elementary equivalence classes of generic models of T are precisely (and irredundantly) given by the classes of generic models of $\bar{J}$ as J ranges over the components of T_U. In particular, $T^F = \cap\{\bar{J}^F\colon J$ is a component of $T_U\}$.*

PROOF. If J is a component of T_U, then by Theorem 3.8 the generic models of $\bar{J}$ are generic models of T. Since J is irreducible, these models are elementarily equivalent by Theorem 3.7.

Conversely, if $\mathcal{M}$ is a generic model of T, then the theory of $\mathcal{M}$ has an irreducible ideal of universal consequences, which by the other half of Theorem 3.8 must be a component of T_U.

Finally, it follows from what we have just shown that elementarily inequivalent generic models of T correspond to distinct components of T_U.

THEOREM 3.10. *Let T_1 and T_2 be inductive. Every generic model of T_1 is a generic model of T_2 if and only if every component of T_{1U} is a component of T_{2U}.*

PROOF. Immediate from Theorem 3.9.

Let us call a family $\mathcal{I}$ of inductive theories *generically regular* if every generic model of $\cap\,\mathcal{I}$ is a generic model of some member of $\mathcal{I}$, or equivalently $\{T_U\colon T \in \mathcal{I}\}$ is a regular family of ideals in $\mathcal{L}_U$.

THEOREM 3.11. *Let $T_{\xi\eta}$, $\langle \xi,\eta\rangle \in \alpha \times \beta$, be a doubly indexed family of inductive theories such that for each $\xi \in \alpha$, $\{T_{\xi,\eta}\colon \eta \in \beta\}$ is generically regular. Then $\mathcal{I} = \{\bigvee_{\xi\in\alpha} T_{\xi.f(\xi)}\colon f \in \beta^\alpha\}$ is generically regular, and $\bigvee_{\xi\in\alpha}\bigvee_{\eta\in\beta} T_{\xi,\eta}$ has the same generic models as $\bigwedge_{f\in\beta^\alpha}\bigvee_{\xi\in\alpha} I_{\xi.f(\xi)}$, i.e., these two theories are mutually model-consistent.*

PROOF. Exactly as in Theorem 1.6, replacing occurences "irreducible ideal containing" by "generic model of".

We conclude with the following problem: in the decomposition $T^F = \cap\{\bar{J}^F\colon J$ is a component of $T_U\}$ of Theorem 3.9 above, does *every* complete extension of T^F occur as one of the $\bar{J}^{F}$'s? The answer is easily seen to be in the affirmative when the decomposition has only finitely many terms or when T^F is model-complete (i.e., T has a model-companion).

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